Angular velocity, angular speed, and acceleration

Example

A car's tachometer records the engine speed is 3000 revolutions per minute. What is this in rads^{-1}?

1 revolution is 2π radians, so we are moving at 6000π radians per minute.
One minute is 60 seconds, so 100 radians per second, ie $314\,\mathrm{rads}^{-1}$.

Example

It takes light 8 minutes and 20 seconds to reach the earth from the sun. How fast is the earth moving?

Speed of light is $\sim 3\times10^{8}\mathrm{ms}^{-1}$. So light travels $3\times10^{8}\times500 = 1.5\times10^{11}\mathrm{m}$, to get to earth.
The distance the sun travels in one year, will therefore be $2\pi r = 9.42\times10^{11}$
One year is $365\times24\times60\times60 = 3.155\times10^{7}\mathrm{s}$.

$$\begin{aligned} speed &= \frac{distance}{time} \\ &= \frac{9.42\times10^{11}}{3.155\times10^{7}} \\ &= 2.98\ldots\times10^{4} \end{aligned}$$

So the speed of the Earth is just shy of $30\,000\mathrm{ms}^{-1}$

Fact — A particle moving in a circle of radius r with angular speed ω has tangential speed v, given by $v = r\omega$.

Example

The pilot of an aircraft flying at $800\,\mathrm{kmh}^{-1}$ on a bearing of 250° receives orders to change course to 210°. The manoeuvre is completed in 20 seconds. Calculate the radius of the turn.

The direction of flight changes 40° in 20 seconds, or 2° per second, this is $\frac{1}{90}\pi\mathrm{rads}^{-1}$. The speed of the aircraft in kms^{-1} which is $\frac{800}{60\times60} = \frac{2}{9}$.

$$\begin{aligned} v &= r\omega \\ \Rightarrow \quad r &= \frac{v}{\omega} \\ &= \frac{\frac{2}{9}}{\frac{\pi}{90}} = \frac{20}{\pi} = 6.37\ldots \end{aligned}$$

Therefore the radius is 6.37 km.

Fact — A particle moving in a circle of radius r with constant angular speed ω and tangential speed v

has an acceleration of magnitude $r\omega^2$, or $\frac{v^2}{r}$, towards the centre of the circle.

$$r\omega^2 = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$$

Example

Copse corner has a radius of 440 m. The fastest drivers take the corner at 290 kmh^{-1}. What acceleration do they experience during the corner?

$$a = \frac{v^2}{r} = \frac{\left(\frac{290\times10^3}{60\times60}\right)^2}{440} = 14.7\,\mathrm{ms}^{-2}$$

People sometimes like to quote this in terms of g, ie $1.5g$.

Example

A particle of 150 g moves in a horizontal circle of radius 50 cm at a constant speed of 4 ms^{-1}. Find the force towards the centre of the circle that must act on the particle.

$$N2: \quad F = ma$$

$$\left(a = \frac{v^2}{r}\right): \quad = m\frac{v^2}{r}$$

$$= .15 \cdot \frac{4^2}{0.5} = 4.8\,\mathrm{N}$$

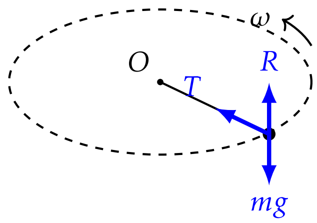
Circular Motion in two dimensions

Example

A particle of mass 300 g is attached to one end of a light in extensible string of length 40 cm, the other endof the string being fixed at O on a smooth horizontal sufrace. If the particle describes circles, centre O, find the tension in the string when

(a) the speed of the particle is $2\sqrt{2}\text{ms}^{-1}$,

(b) the angular speed of the particle is $5\,\text{rads}^{-1}$



(a)

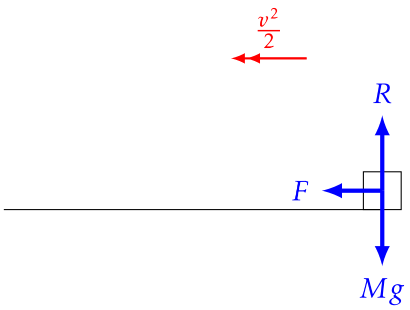
$$\begin{aligned} T &= \frac{mv^2}{r} \\ &= \frac{0.3(2\sqrt{2})^2}{0.4} \\ &= 6\,\text{N} \end{aligned}$$

(b)

$$\begin{aligned} T &= m\frac{v^2}{r} = mr\omega^2 \\ &= 0.3 \cdot 0.4 \cdot 5^2 \\ &= 3\,\text{N} \end{aligned}$$

Example

A car of mass Mkg is travelling round a bend which is an arc of a circle radius 140 m. The greatest speed at which the car can travel round the bend without slipping is 45 kmh^{-1}. Find the coefficient of friction between the tyres of the car and the road.



Let's assume the car is travelling at 45 kmh^{-1} so we are at the point of slipping, ie $F = F_{max}$. Then

$$\begin{aligned}
N2(\uparrow): && R - Mg &= 0 \\
\Rightarrow && R &= Mg \\
N2(\leftarrow, radially): && F_{max} &= \frac{mv^2}{r} \\
&& &= \frac{M \cdot \left(\frac{45\times1000}{60\times60}\right)^2}{140} \\
&& &= M\frac{125}{112} \\
&& F_{max} &= \mu R \\
&& &= \mu Mg \\
\Rightarrow && \mu &= \frac{F_{max}}{Mg} \\
&& &= \frac{M\frac{125}{112}}{M \cdot 9.8} \\
&& &\approx 0.11
\end{aligned}$$

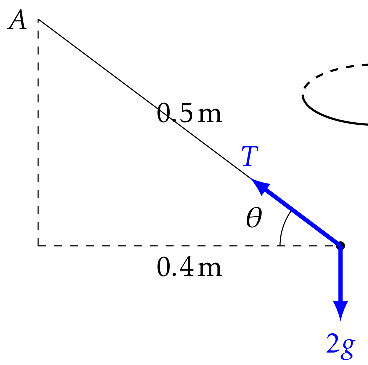
Therefore the coefficient of friction is 0.11.

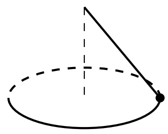
Horizontal Circular Motion in three dimensions

Conical Pendulum

Example

A particle of mass 2 kg is attached to one end of a light inextensible string of length 50 cm. The other end of the string is attached to a fixed point A. The particle moves with constant angular speed in a horizontal circle of radius 40 cm. The centre of the circle is vertically below A. Calculate the tension in the string and the angular speed of the particle.





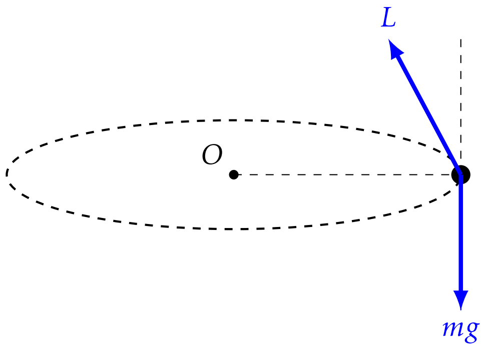
Notice this is a $3-4-5$ *triangle, so* $\cos\theta = \frac{3}{5}, \sin\theta = \frac{4}{5}$

$$
\begin{aligned}
N2(\uparrow): &\quad T\sin\theta - 2g = 0\\
\Rightarrow &\quad T\sin\theta = 2g\\
\Rightarrow &\quad T = \frac{2g}{\frac{3}{5}} = 32.66\ldots\\
&\quad = 32.7\,\text{N}\\
N2(\leftarrow, radially): &\quad T\cos\theta = mr\omega^2\\
\Rightarrow &\quad \omega^2 = \frac{T\cos\theta}{mr}\\
&\quad = \frac{T\cdot 0.8}{2\cdot 0.4}\\
&\quad = 32.66\ldots\\
\Rightarrow &\quad \omega = 5.7\,\text{rads}^{-1}
\end{aligned}
$$

Banked Corner

Example

An aircraft of mass 2 tonnes flies at 500 $\text{km}\,\text{h}^{-1}$ on a path which follows a horizontal circular arc in order to change course from due north to due east. The aircraft turns in the clockwise direction from due north to due east. It takes 40 s to change course, with the aircraft banked at an angle α to the horizontal. Calculate the value α and the magnitude of the lift force perpendicular to the surface of the aircraft's wings.



(a) Finding the radius:
The aircraft turns through 90° in 40 seconds.
Angular velocity:

$$\omega = \frac{90^\circ \times \frac{\pi}{180}}{40} = \frac{\pi}{80} \; rad\ s^{-1}$$

Speed:

$$v = \frac{500 \times 1000}{60 \times 60} = \frac{1250}{9} = 138.88\ldots \; m\ s^{-1}$$

Using $v = r\omega$:

$$\begin{aligned} r &= \frac{v}{\omega} \\ &= \frac{1250/9}{\pi/80} \\ &= \frac{1250 \times 80}{9\pi} \\ &= \frac{100000}{9\pi} \\ &= 3537.0\ldots \; m \end{aligned}$$

Therefore the radius is 3540 m (3 s.f.)
(b) Finding the banking angle:

$$\begin{aligned} N2(\uparrow): &\quad L\cos\alpha - mg = 0 \\ \Rightarrow &\quad L\cos\alpha = mg \\ N2(\leftarrow, radially): &\quad L\sin\alpha = m\frac{v^2}{r} \\ \Rightarrow &\quad \tan\alpha = \frac{v^2}{rg} \end{aligned}$$

$$= \frac{138.88\ldots^2}{3537.0\ldots \cdot 9.81}$$
$$= 0.5559\ldots$$
$$\Rightarrow \qquad \alpha = 29.1°$$

Therefore the banking angle is $\boxed{29°}$ *(to the nearest degree).*
(c) Finding the lift force:
From equation (1):

$$\begin{aligned} L &= \frac{mg}{\cos\alpha} \\ &= \frac{2000 \times 9.8}{\cos(29.1°)} \\ &= \frac{19600}{0.8733...} \\ &= 22445...\ N \end{aligned}$$

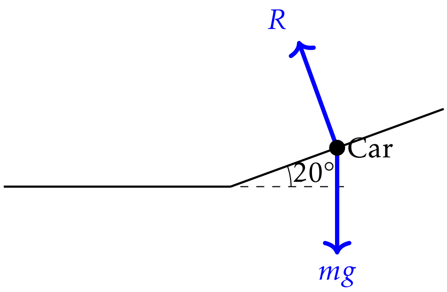
Therefore the lift force is 22 400 N *(3 s.f.) or* 22.4 kN.

Example

A racing track has a circular bend of radius 150 m banked at an angle of 20° to the horizontal. A car of mass 800 kg travels around this bend. Find:

(a) the speed at which the car experiences no sideways friction

(b) the normal reaction on the car at this speed



(a)

$$\text{N2}(\uparrow): \quad R\cos 20^\circ - mg = 0$$

$$\text{N2}(\leftarrow): \quad R\sin 20^\circ = m\frac{v^2}{r}$$

$$\Rightarrow \quad \tan 20^\circ = \frac{v^2}{rg}$$

$$\Rightarrow \quad v = \sqrt{150 \cdot 9.8 \cdot \tan 20^\circ}$$

$$= 23.1\,\text{ms}^{-1}$$

(b)

$$R = \frac{mg}{\cos 20^\circ}$$

$$= \frac{800 \cdot 9.8}{\cos 20^\circ}$$

$$= 8343.1\ldots$$

$$= 8.3\,\text{kN}$$

Example (OCR M2 January 2010 Q7)

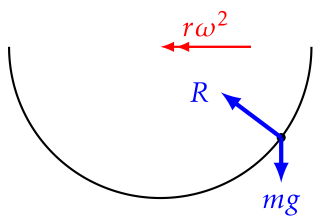
A particle P of mass 0.2 kg is moving on the smooth inner surface of a fixed hollow hemisphere which has centre O and radius 5 m. P moves with constant angular speed ω in a horizontal circle at a vertical distance of 3 m below the level of O. (See Fig. 1)

(i) Calculate the magnitude of the force exerted by the hemisphere on P [3]

(ii) Calculate ω [4]

A light inextensible string is how attached to P. The string passes through a small hole at the end of the lowest point of the hemisphere and a particle of mass 0.1 kg hangs in equilibrium at the end of the string. P moves in the same horizontal circle as before (see Fig. 2)

(iii) Calculate the new angular speed of P [8]

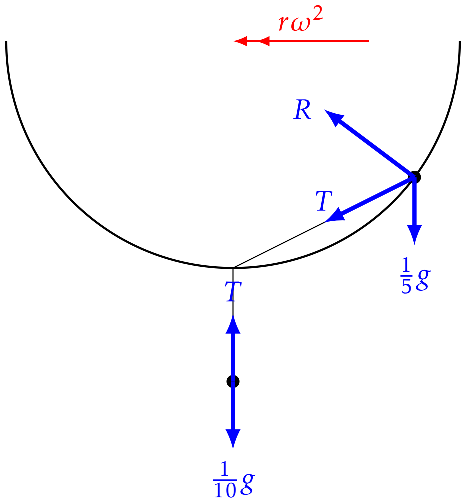


(i)

$$\text{N2}(\uparrow): \quad R\cos\theta - mg = 0$$
$$\Rightarrow \quad \frac{3}{5}R = 0.2g$$
$$\Rightarrow \quad R = \frac{1}{3}g = 3.27\,\text{N (3 s.f.)}$$

(ii)

$$\text{N2}(\leftarrow, \text{radially}): \quad R\sin\theta = mr\omega^2$$
$$\Rightarrow \quad \omega^2 = \frac{\frac{4}{5}\frac{1}{3}g}{0.2\cdot 4} = \frac{1}{3}g$$
$$\Rightarrow \quad \omega = \sqrt{3.27\ldots} = 1.81\,\text{rads}^{-1}$$



$$
\begin{aligned}
N2(\uparrow, new\ particle): \quad & T - \frac{1}{10}g = 0 \\
\Rightarrow \quad & T = \frac{g}{10} \\
\\
N2(\uparrow, P): \quad & R\cos\theta - T\cos\alpha - \frac{1}{5}g = 0 \\
\Rightarrow \quad & \frac{3}{5}R - \frac{2}{\sqrt{5}}T - \frac{1}{5}g = 0 \\
\Rightarrow \quad & \frac{3}{5}R = \frac{2}{\sqrt{5}}\frac{g}{10} + \frac{g}{5} \\
\Rightarrow \quad & R = \frac{g}{3}\left(1 + \frac{2}{\sqrt{5}}\right) \\
\\
N2(\leftarrow, P, radially): \quad & \frac{4}{5}R + \frac{1}{\sqrt{5}}T = mr\omega^2 \\
& \frac{4}{5}\frac{g}{3}\left(1 + \frac{2}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}}\frac{g}{10} = 0.2 \cdot 0.4 \cdot \omega^2 \\
\Rightarrow \quad & \omega^2 = \frac{g}{0.08}\left(\frac{4}{15}\left(1 + \frac{2}{\sqrt{5}}\right) + \frac{1}{10\sqrt{5}}\right) \\
\Rightarrow \quad & \omega \approx 7.2\ rad\ s^{-1}
\end{aligned}
$$